# Sets and Functions

Abstract Algebra = Study of sets with extra structure

Set = collection of objects (or elements of the set)

Examples : $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, M_n(\mathbb{R}), GL_n(\mathbb{R}), \mathbb{R}^n, \ldots$

## Important Notation

should be read as "such that"

$S = \{ \text{Notation for elements} \mid \text{Defining properties} \}$

For example,

contained in — there exists

Even integers $= \{ x \in \mathbb{Z} \mid \exists y \in \mathbb{Z} \text{ such that } x = 2y \}$

for all — y divides x — implies

Prime numbers $= \{ x \in \mathbb{N} \mid \forall y \in \mathbb{N},\ y \mid x \Rightarrow y = x \text{ or } y = 1 \}$

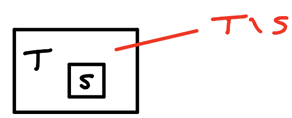
$S, T$ – two sets

subset — if and only if

$S \subset T \iff x \in S \Rightarrow x \in T$

not in

If $S \subset T$, then $T \setminus S = \{ x \in T \mid x \notin S \}$

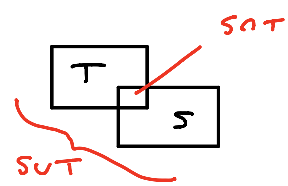



union

$S \cup T = \{ x \mid x \in S \text{ or } x \in T \}$

intersection

$S \cap T = \{ x \mid x \in S \text{ and } x \in T \}$




$S, T$ disjoint $\iff S \cap T = \emptyset$ ← empty set

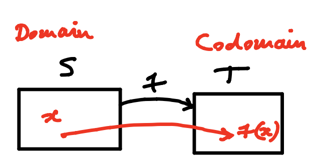
Cartesian product

$S \times T = \{ (s,t) \mid s \in S \text{ and } t \in T \}$

$S^n = \underbrace{S \times S \times \ldots \times S}_{n \text{ times}}$, where $n \in \mathbb{N}$

$f : S \longrightarrow T$ ← Function / map from $S$ to $T$. A rule which associates to any $x \in S$ a unique element $f(x) \in T$

$x \longmapsto f(x)$




$\mathrm{Im}(f) = \{ y \in T \mid \exists x \in S \text{ such that } f(x) = y \}$ ← often called Range of $f$

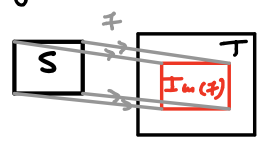
$f$ surjective $\iff \mathrm{Im}(f) = T$

often called "onto".

$f$ <u>injective</u> $\iff$ $f(x) = f(y) \Rightarrow x = y \quad \forall x, y \in S$

often called "one-to-one".

$f$ injective $\Rightarrow$ Can identify $S$ with $Im(f)$




$f$ <u>bijective</u> $\iff$ $f$ injective <u>and</u> $f$ surjective

$f$ bijective $\Rightarrow$ Can identify $S$ with $T$ using $f$.

size / cardinality

$f : S \to T$ bijection and $|S| < \infty$, $|T| < \infty$ $\Rightarrow$ $|S| = |T|$

Important Exercise :

$f : S \to T$ bijection $\iff$ $\exists g : T \to S$ such that

composition $(f \circ g)(x) = f(g(x))$

$f \circ g = Id_T$ <u>and</u> $g \circ f = Id_S$

$Id_S(x) = x \ \forall x \in S$

Identity function

## <u>Equivalence Relations</u>

<u>Definition</u> A <u>partition</u> of a set $S$ is a collection of subsets $S_i$

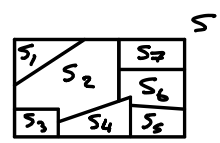
$i$ in some index set $I$

such that

pairwise disjoint

- $S_i \cap S_j \neq \emptyset \quad \forall i \neq j$
- $\coprod_{i \in I} S_i = S$

disjoint union




<u>Example</u> $S = \mathbb{Z}$, $S_1$ = even integers , $S_2$ = odd integers.

<u>Definition</u> A <u>relation</u> on a set $S$ is a subset $\mathcal{U} \subset S \times S$

If $\mathcal{U} \subset S \times S$ is a relation we write

"$x$ related to $y$"

$x \sim y \iff (x, y) \in \mathcal{U}$.

<u>Definition</u> An <u>equivalence relation</u> on a set $S$ is a relation $\sim$ such that

- $x \sim x \quad \forall x \in S$ (reflexive)
- $x \sim y \iff y \sim x \quad \forall x, y \in S$ (Symmetric)
- $x \sim y$, $y \sim z \Rightarrow x \sim z$ (Transitive)

<u>Definition</u> If $\sim$ is an equivalence relation on a set $S$ and $x \in S$

equivalence class containing $x$

$[x] := \{ y \in S \mid y \sim x \} \subset S$

<u>Proposition</u> If $\sim$ is an equivalence relation on a set $S$, then the equivalence classes form a partition of $S$.

<u>Proof</u>

$x \sim x \Rightarrow x \in [x], \forall x \in S \Rightarrow$ union of equivalence classes is $S$

$y \in [x] \Rightarrow y \sim x \Rightarrow [y] \subset [x]$

$z \in [x] \cap [y] \Rightarrow z \sim x, z \sim y \Rightarrow x \sim z, z \sim y$

$\Rightarrow x \sim y \Rightarrow y \sim x \Rightarrow [x] \subset [y]$ and $[y] \subset [x]$

$\Rightarrow [x] = [y]$

Hence equivalence classes are equal or disjoint.

$\square$

<u>Important Exercise</u> : Given <u>any</u> partition $S_i$ of $S$ there exists a <u>unique</u> equivalence relation $\sim$ whose equivalence classes are the $S_i$.

<u>Conclusion</u> : Partition of $S$ = Equivalence relation on $S$

Very important